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An elemental characterization of strong primeness in Jordan systems

José A. Anquela[a,*], Teresa Cortés[a], Ottmar Loos[b], Kevin McCrimmon[c]

[a]*Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, C Calvo Sotelo s/n, 33007 Oviedo, Spain*

[b]*Institut für Mathematik, Universität Innsbruck, A-6020 Innsbruck, Austria*

[c]*Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903, USA*

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Abstract

We give elemental characterizations of strong primeness for Jordan algebras, pairs and triple systems. We use our characterization to study the transfer of strong primeness between a Jordan system and its local algebras and subquotients.

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0. Introduction

A ring R is called prime if it contains no nonzero ideals annihilating each other. It is easy to see that this is equivalent to the elemental condition that $aRb = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in R$. The analogous notion of primeness for Jordan algebras is much less satisfying, mainly due to the difficulty of building ideals. Unlike the associative case, primeness does not imply the absence of absolute zero divisors and therefore leads to the definition of strong primeness. An elemental characterization, analogous to the associative case, was given for linear Jordan algebras in [2], namely, $U_{a,b} = 0$ implies $a = 0$ or $b = 0$. But this characterization does not carry over to quadratic Jordan algebras, as shown by the example of a field of characteristic two. Moreover, it involves the bilinear operators $U_{a,b}$ which do not enjoy the multiplicative properties of their quadratic counterparts. In this paper, we provide elemental characterizations of strong primeness for all three types of Jordan systems (algebras, triples and pairs) over an arbitrary ring of scalars in terms of the quadratic multiplication operators, as follows:

*Corresponding author.

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- A Jordan pair (V^+, V^-) is strongly prime if and only if $Q_a Q(V^{\mp}) Q_b = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in V^{\pm}$ (Theorem 1.10);
- a Jordan algebra J is strongly prime if and only if $U_a U_J U_b = 0$ implies $a = 0$ or $b = 0$ (Theorem 1.13);
- a Jordan triple system T is strongly prime if and only if $P_a P_T P_b = P_a P_T P_T P_b = 0$ implies $a = 0$ or $b = 0$ (Theorem 2.9).

While the characterizations of strong primeness for pairs and algebras are formally the same (and indeed a Jordan algebra J is strongly prime if and only if the associated Jordan pair (J, J) is so), Jordan triple systems seem to behave differently. This is explained by the fact that a Jordan triple system T can be considered as a Jordan pair $W = (T, T)$ with involution $*$, and then strong primeness of T becomes strong $*$-primeness of W, in exact analogy with the notion of primeness in the category of rings with involution.

The proofs of the elemental characterizations are not elementary, since they make use of the structure theory of strongly prime Jordan pairs [3, 4]: Every such pair is either of hermitian type, containing an ideal which is an ample subpair of a $*$-prime associative pair, or has a tight scalar extension which is simple and equals its socle. Accordingly, we prove the elemental characterization in these cases separately (Lemmas 1.8 and 1.9) and then use an up-and-down argument (Proposition 1.6) to settle the general case. The result for algebras follows easily from the one for pairs, while the proof for triple systems requires more effort. It reduces to the pair case by another up-and-down argument (Proposition 2.8) based on the usual dichotomy for strongly prime systems: Either the Jordan pair $W = (T, T)$ is strongly prime, or W contains a strongly prime ideal I with $I \cap I^* = 0$.

In a final section we consider the local algebras [5] and subquotients [9] of Jordan systems. Using the elemental characterization we show that a Jordan algebra or pair is strongly prime if and only if all subquotients and local algebras are so (Theorem 3.2, Corollary 3.3). In the triple case the local algebras are either strongly prime or a subdirect product of two strongly prime algebras, and local algebras of the first type always exist (Theorem 3.5).

All Jordan systems considered are modules over an arbitrary commutative ring of scalars. Our basic references regarding notation and terminology are [6] and [7].

1. Elemental primeness in Jordan pairs and algebras

1.1. We begin dealing with Jordan pairs $V = (V^+, V^-)$ with products $Q_x y \in V^\varepsilon$, for $x \in V^\varepsilon$, $y \in V^{-\varepsilon}$, which are linear in y and quadratic in x. Recall that V is nondegenerate if it has no nonzero trivial elements; i.e., $Q_x = 0$ implies $x = 0$. Then any nonzero ideal $I = (I^+, I^-)$ has $I^+ \neq 0 \neq I^-$, because I inherits nondegeneracy from V by [7, 4.13]. We also recall that an ideal I is *orthogonal* to an ideal L in a Jordan pair, $I \perp L$, if $Q_{I^\varepsilon} L^{-\varepsilon} = 0$, for $\varepsilon = \pm$. A Jordan pair is called *prime* when it has no nonzero orthogonal ideals, and *strongly prime* if it is prime and nondegenerate. Strong primeness for a Jordan pair V is equivalent to the condition that V be nondegenerate and ideals satisfy the finite intersection property.

1.2. Definition. Let V be a Jordan pair. An element $a \in V^\varepsilon$ ($\varepsilon = \pm$) is called *quadratically orthogonal* (q-orthogonal for short) to $b \in V^\varepsilon$, written $a] [b$ or $a] [_V b$, when it is necessary to emphasize the dependence on V, if $Q_a Q_{V^{-\varepsilon}} Q_b = 0$.

Note that for any pair of ideals I, L of a Jordan pair V orthogonality $I \perp L$ immediately implies that any $a \in I^\varepsilon$ is q-orthogonal to any $b \in L^\varepsilon$ ($\varepsilon = \pm$). The converse is true when the pair is nondegenerate: If any element in I^ε is q-orthogonal to any element in L^ε ($\varepsilon = \pm$), then for any $a \in I^\varepsilon$, $b \in L^{-\varepsilon}$, $x \in V^{-\varepsilon}$, the element $c = Q(Q_a b)x$ has

$$Q_c V^{-\varepsilon} = Q(Q_a b) Q_x Q(Q_a b) V^{-\varepsilon} \subset Q(I^\varepsilon) Q(V^{-\varepsilon}) Q(L^\varepsilon)\, V^{-\varepsilon} = 0,$$

which implies $c = Q(Q_a b)x = 0$, hence $Q_a b = 0$ since V is nondegenerate.

A Jordan pair V will be called *elementally prime* if neither V^+ nor V^- contains nonzero elements a, b such that a is q-orthogonal to b. Thus by definition, V is elementally prime if and only if $V^{\mathrm{op}} = (V^-, V^+)$ is so. However, we will see later (Theorem 1.10) that in the presence of nondegeneracy, the above condition for $\varepsilon = +$ is sufficient for elemental primeness.

It is immediate that any elementally prime Jordan pair is nondegenerate, so that the above remarks readily imply the following.

1.3. Lemma. *An elementally prime Jordan pair is strongly prime.*

Our aim is to prove the converse of the above result. We begin with the study of the transfer of elemental primeness between a Jordan pair and its nonzero ideals.

Recall that a structural transformation $(f, f^*): U \rightleftarrows W$ between Jordan pairs is a pair of linear maps $f: U^+ \to W^+$, $f^*: W^- \to U^-$ satisfying $Q_{f(x)} = f Q_x f^*$ and $Q_{f^*(y)} = f^* Q_y f$ for all $x \in U^+$ and $y \in W^-$. We will be mostly interested in the case where U and W are either V or V^{op} for some Jordan pair V and (f, f^*) is built out of the inner structural transformations (Q_x, Q_x).

1.4. Lemma. *Let $(f, f^*): U \rightleftarrows W$ be a structural transformation between nondegenerate Jordan pairs. Then*

(a) $f = 0$ *implies* $f^* = 0$;

(b) $f Q_x = 0$ *implies* $f(x) = 0$;

(c) *If f vanishes on a nonzero ideal I of U ($f(I^+) = 0$) with $f^*(W^-) \subset I^-$ (e.g. if f is inner structural with a factor from I) then $f = 0$.*

Proof. (a) All $f^*(y)$ are trivial, $Q_{f^*(y)} = f^* Q_y f = 0$, and hence 0 (this can also be obtained as a consequence of [9, 1.10]).

(b) $f Q_x = 0$ implies $Q_{f(x)} = f Q_x f^* = 0$ and therefore $f(x) = 0$.

(c) For any $x \in U^+$, $Q_{f(x)} W^- = fQ_x f^*(W^-) \subset fQ_x I^- \subset f(I^+) = 0$, which implies $f = 0$ by nondegeneracy. $\square$

1.5. Lemma. *In a nondegenerate Jordan pair V, the relation $a] \, [b$ (for $a, b \in V^\varepsilon$) is symmetric in a and b, and implies*

$$Q_a Q(V^{-\varepsilon}) b = Q_a D(V^{-\varepsilon}, b) = Q(a, b) = 0.$$

Proof. Symmetry follows from Lemma 1.4(a) for $(f, f^*) = (Q_a Q_x Q_b, Q_b Q_x Q_a)$, where $x \in V^{-\varepsilon}$. If $a] \, [b$ then $Q_a Q_{V^{-\varepsilon}} b = 0$ by Lemma 1.4(b) applied to $(f, f^*) = (Q_a Q_x, Q_x Q_a)$, for $x \in V^{-\varepsilon}$. Hence $0 = Q_a Q(V^{-\varepsilon}, V^{-\varepsilon}) b = Q_a D(V^{-\varepsilon}, b) V^{-\varepsilon}$ gives $Q_a D(V^{-\varepsilon}, b) = 0$. To finish the proof, we have $Q(a, b) V^{-\varepsilon} = 0$ since $\{axb\}$ is trivial for any $x \in V^{-\varepsilon}$: By the identity JP21 of [7], $Q(\{axb\}) = Q_a Q_x Q_b + Q_b Q_x Q_a + D(a, x) Q_b D(x, a) - Q(Q_a Q_x b, b) = 0$ by symmetry and the equalities already proved. $\square$

1.6. Proposition. *Let V be a Jordan pair and I a nonzero ideal of V.*

(a) *If $a, b \in I^\varepsilon$ is nondegenerate then $a] \, [_I b$ implies $a] \, [_V b$. Hence, if V is elementally prime so is I.*

(b) *Conversely, if I is elementally prime and V is tight extension of I (in the sense that any nonzero ideal of V intersects I nontrivially) then V is elementally prime. Hence, if V is prime and I is elementally prime then V is elementally prime.*

Proof. (a) Possibly after replacing V by V^{op} we may assume $\varepsilon = -$. Thus let $a, b \in I^-$ satisfy $Q_a Q(I^+) Q_b I^+ = 0$ and let $x \in V^+$. By Lemma 1.4(c), applied to $(f, f^*) = (Q_a Q_x Q_b, Q_b Q_x Q_a)$, to show $f = 0$ it is enough to prove that $f(I^+) = 0$. Now for any $y \in I^+$,

$$\begin{aligned} Q_{f(y)} V^+ &= Q_a Q_x Q_b Q_y Q_b Q_x Q_a V^+ \subset Q_a Q_x Q_b Q_y Q_b I^+ \\ &= Q_a(-Q_y Q_b Q_x + Q_{\{xby\}} + Q(Q_x Q_b y, y) - D_{x,b} Q_y D_{b,x}) Q_b I^+ \\ &\subset Q_a Q_{I^+} Q_b Q_x Q_b I^+ + Q_a Q_{I^+} Q_b I^+ + Q_a Q_{I^+, I^+} Q_b I^+ \\ &\quad + Q_a D_{x,b} Q_y D_{b,x} Q_b I^+, \end{aligned}$$

by the identity JP21 of [7]. Here the first three terms on the right-hand side vanish since $Q_x Q_b I^+ \subset I^+$ and $a] \, [_I b$, and the last term vanishes similarly since

$$\begin{aligned} Q_a D_{x,b} Q_y D_{b,x} Q_b I^+ &\subset Q_a D_{x,b} Q_y Q_b D_{x,b} I^+ \subset Q_a D_{x,b} Q_y Q_b I^+ \\ &= Q_a(-Q_y D_{b,x} + Q(y, \{xby\})) Q_b I^+ \\ &\subset Q_a Q_y Q_b D_{x,b} I^+ + Q_a Q(y, \{xby\}) Q_b I^+ \\ &\subset Q_a Q_{I^+} Q_b I^+ + Q_a Q_{I^+, I^+} Q_b I^+ = 0. \end{aligned}$$

Here we used the identifies JP1, JP12, and again JP1. Since V is nondegenerate, $f(y) = 0$, as desired.

(b) First, V is nondegenerate. Indeed, if rad denotes the lower radical, then $0 = \operatorname{rad} I = I \cap \operatorname{rad} V$ [7, 4.13] implies $\operatorname{rad} V = 0$ by tightness. Also, as V is tight, primeness of I implies primeness of V. Now let $a, b \in V^{\varepsilon}$ be q-orthogonal and $y, z \in I^{-\varepsilon}$. Then the elements $a_1 = Q_a y$, $b_1 = Q_b z$ lie in I^{ε} and satisfy

$$Q_{a_1}, Q_{I^{-\varepsilon}} Q_{b_1} I^{-\varepsilon} \subset Q_a Q_y Q_a Q_{V^{-\varepsilon}} Q_b Q_z Q_b V^{-\varepsilon} = 0.$$

Elemental primeness of I implies that either $Q_a I^{-\varepsilon} = 0$ or $Q_b I^{-\varepsilon} = 0$. From nondegeneracy of V and [11, 1.7] it follows that a or b belongs to the annihilator of I which is zero by primeness. □

1.7. Ample subpairs. The following lemma deals with Jordan pairs which are ample subpairs of associative pairs. Recall that given an associative pair $R = (R^+, R^-)$, the underlying modules of R become a Jordan pair with quadratic operators $Q_x y = xyx$, where juxtaposition denotes the associative products in R. A polarized involution $*$ of R is a pair of linear maps $*: R^{\varepsilon} \to R^{\varepsilon}$ of period two such that $(xyz)^* = z^* x^* y^*$. By an *ample subpair* of $(R, *)$ we mean a pair $V = (V^+, V^-)$ of submodules $V^{\varepsilon} \subset R^{\varepsilon}$ containing all traces and contained in the symmetric elements,

$$\{x + x^*: x \in R^{\varepsilon}\} \subset V^{\varepsilon} \subset \{x: x = x^*\},$$

and with the property that

$$x V^{-\varepsilon} x^* \subset V^{\varepsilon} \quad \text{for any } x \in R^{\varepsilon}.$$

It is easy to see that ample subpairs are Jordan subpairs of R, and that the traces and the symmetric elements form the smallest and the largest ample subpair, respectively.

1.8. Lemma. *Let R be an associative pair with polarized involution $*$. If an ample subpair V of $(R, *)$ is strongly prime then it is elementally prime.*

Proof. By factoring out a maximal $*$-ideal of R intersecting V trivially we can assume that R is a $*$-tight cover of V, hence R is $*$-prime.

Let $0 \neq a, b \in V^+$. If $a] [b$ then $Q_{a,b} = 0$ by Lemma 1.5. For any $x \in V^-$,

$$(Q_a x) V^- (Q_b x) = 0. \tag{1}$$

Indeed, for any $y \in V^-$, $(Q_a x) y (Q_b x) = axaybxb = ax(Q_{a,b} y)xb - axbyaxb = -axbyaxb = -(Q_{a,b} x) yaxb + bxayaxb = bxayaxb = Q_b Q_x Q_a y = 0$ by Lemma 1.5. Next we claim that there exists $x_0 \in V^-$ such that $a_1 = Q_a x_0$ and $b_1 = Q_b x_0$ are both $\neq 0$. Otherwise, $V^- = \operatorname{Ker} Q_a \cup \operatorname{Ker} Q_b$, which implies, since the latter are submodules, that either $V^- = \operatorname{Ker} Q_a$ or $V^- = \operatorname{Ker} Q_b$, contradicting the fact that V is nondegenerate. By (1),

$$a_1 V^- b_1 = 0. \tag{2}$$

As above, we can find $x_1 \in V^-$ such that $a_2 = Q_{a_1} x_1 \neq 0$, $b_2 = Q_{b_1} x_1 \neq 0$. We claim that

$$a_2 R^- b_2 = 0. \tag{3}$$

Indeed, for any $r \in R^-$, $a_2 r b_2 = a_1 x_1 a_1 r b_1 x_1 b_1 = a_1 x_1 (a_1 r b_1 + (a_1 r b_1)^*) x_1 b_1 - a_1 x_1 b_1 r^* a_1 x_1 b_1 = 0$, because the first summand is $a_1 (Q_{x_1}(a_1 r b_1 + (a_1 r b_1)^*))\, b_1 \subset a_1 (Q_{x_1} V^+) b_1$ (since V is an ample subpair) $\subset a_1 V^- b_1 = 0$ by (2), while the second summand is zero by (2). But (3) contradicts the fact that R is $*$-prime, since it clearly implies that the nonzero $*$-ideals generated by a_2 and b_2 are orthogonal. The same argument works with $+$ and $-$ interchanged, so the proof is complete. □

1.9. Lemma. *Any simple nondegenerate Jordan pair equal to its socle (equivalently, with dcc on principal inner ideals) is elementally prime.*

Proof. Let V be such a Jordan pair, let $a, b \in V^\varepsilon$ be q-orthogonal and assume $b \neq 0$. Since V^{op} satisfies the same hypotheses as V it is no restriction to assume $\varepsilon = +$. It is known that V is von Neumann regular [8, Theorem 1]. Hence $b = e_+$ can be completed to a nonzero idempotent $e = (e_+, e_-)$ of V [7, 5.2]. Denote $V = V_2 \oplus V_1 \oplus V_0$ the Peirce decomposition of V with respect to e. It is readily seen from Peirce multiplication rules [7, 5.4] that $Q(V^-) V_2^+ = V_2^- \oplus V_1^- \oplus Q(V_1^-) V_2^+$. By [10, 2.14] and simplicity, $V_0^- = Q(V_1^-) V_2^+$ so that $Q(V^-) V_2^+ = V^-$. Also, $V_2^+ = Q_{e_+} V^- = Q_b V^-$. Thus the condition $a][b$ becomes $0 = Q_a Q(V^-) Q_b V^- = Q_a Q(V^-) V_2^+ = Q_a V^-$ and implies $a = 0$ by nondegeneracy. □

1.10. Theorem. *The following conditions on a Jordan pair V are equivalent.*

(i) *V is strongly prime;*
(ii) *V is elementally prime;*
(iii) *V is nondegenerate and V^+ contains no nonzero q-orthogonal elements;*
(v) *V is nondegenerate and V^- contains no nonzero q-orthogonal elements.*

Proof. (ii) ⇒ (i) is Lemma 1.3.

(i) ⇒ (ii): By [3] and [4] either V has a tight scalar extension $\tilde{V}$ which is simple with nonzero socle, or V has a nonzero ideal I which is an ample subpair of some associative pair R with polarized involution. Any V of the first kind is elementally prime, since $a]\,[_V b$, $a, b \in V^\varepsilon$, readily implies $a]\,[_{\tilde{V}} b$, and therefore $a = 0$ or $b = 0$ by Lemma 1.9. If V is of the second kind then I is strongly prime by [11], hence elementally prime by Lemma 1.8, which implies that V is elementally prime by Proposition 1.6(b).

(ii) ⇒ (iii) is immediate since elemental primeness implies nondegeneracy by Lemma 1.3.

(iii) ⇒ (i): If two ideals I, L of V are orthogonal then any element in I^+ is q-orthogonal to any element in L^+, by Definition 1.2. Hence (iii) implies that either $I^+ = 0$ or $L^+ = 0$, and therefore $I = 0$ or $L = 0$ by 1.1 since V is nondegenerate.

We have shown that (i)–(iii) are equivalent. Their equivalence with (iv) follows from the symmetry in + and − of (i) and (ii). □

1.11. We immediately get an analogue of Theorem 1.10 for Jordan algebras. As usual, the quadratic operators and squaring of a Jordan algebra J are denoted by U_x and x^2. Nondegeneracy, orthogonality of ideals and (strong) primeness for Jordan algebras are defined as for Jordan pairs, with obvious changes. Recall that any Jordan algebra J gives rise to the Jordan pair (J, J), with quadratic operators $Q_x = U_x$, which is obviously nondegenerate if and only if J is so.

Two elements a, b in Jordan algebra J are called q-orthogonal if $U_a U_J U_b = 0$, i.e., if they are so in (J, J), and J is said to be elementally prime if J has no nonzero q-orthogonal elements, that is to say, if (J, J) is elementally prime.

1.12. Lemma. *A Jordan algebra J is strongly prime if and only if (J, J) is strongly prime.*

Proof. Let (J, J) be strongly prime. Then J cannot have two nonzero orthogonal ideals I, L for they would give rise to nonzero orthogonal ideals (I, I), (L, L) of (J, J). Thus J is prime.

Conversely, assume that J is strongly prime. Since the lower radical of a Jordan algebra is hereditary [7, 4.13, 4.17], any tight unital hull $\hat{J}$ of J is strongly prime. Now $(\hat{J}, \hat{J})$ is strongly prime by [7, 1.6], therefore (J, J) is strongly prime by [11] since it is an ideal of $(\hat{J}, \hat{J})$. □

Now Theorem 1.10, 1.11 and Lemma 1.12 yield:

1.13. Theorem. *A Jordan algebra is strongly prime if and only if it is elementally prime.*

2. Elemental primeness in Jordan triple systems

2.1. Now we turn to the case of Jordan triple systems T which, compared to pairs and algebras, presents some new features. We denote the quadratic operators in T by P_x. Recall [7, 1.13] that any Jordan triple system T gives rise to a Jordan pair $W = (T, T)$ with quadratic operators $Q_x = P_x$ and involution $*: W \to W^{\mathrm{op}}$ given by the identity map on T. This establishes an equivalence between the categories of Jordan triple systems and Jordan pairs with involution. Also, every Jordan pair $V = (V^+, V^-)$ defines a polarized Jordan triple system $V^+ \oplus V^-$, and in this way the category of Jordan pairs is equivalent to that of polarized Jordan triple systems [7, 1.14]. Depending on the context, one or the other point of view will be more convenient. It is clear that nondegeneracy is preserved under these equivalences.

As expected, a Jordan triple system T is called *strongly prime* if it is nondegenerate, and has no nonzero orthogonal (or disjoint) ideals. Strong primeness of T is equivalent to the condition that the Jordan pair $W = (T, T)$ be strongly $*$-prime in the sense that it is nondegenerate and has no nonzero orthogonal $*$-ideals. Note, however, that W need not be prime. This happens, for instance, when $T = V^+ \oplus V^-$ is the polarized Jordan triple system of a strongly prime Jordan pair V: According to the following lemma, T is strongly prime but (T, T) is the direct sum of the ideals $I = (V^+ \oplus 0, 0 \oplus V^-)$ and I^*.

2.2. Lemma. *A Jordan pair $V = (V^+, V^-)$ is strongly prime if and only if the polarized Jordan triple system $T = V^+ \oplus V^-$ is strongly prime.*

Proof. Let T be nondegenerate and let L be a nonzero ideal of T. It is easy to see that $I = (L \cap V^+, L \cap V^-)$ is an ideal of V. Also, $I \neq 0$ for otherwise, by polarization of T, we would have $P_L V^{-\varepsilon} \subset L \cap V^{\varepsilon} = 0$ and thus $L = 0$ by nondegeneracy. Now it is clear that V prime implies T prime, and the converse holds as well because any ideal (I^+, I^-) of V gives rise to the polarized ideal $I^+ \oplus I^-$ of T. $\square$

2.3. *Semi-ideals.* Let W be a Jordan pair. A $(+)$-*ideal* of W is an inner ideal $S \subset W^+$ such that $\{W^+ W^- S\} + Q(W^+)Q(W^-)S \subseteq S$. The notion of $(-)$-ideal is analogous. Clearly, the $+$-part of an ideal of $W$is a $(+)$-ideal. Conversely, it can be shown by a lengthy but straightforward calculation that $(S, Q(W^-)S)$ is an ideal whenever S is a $(+)$-ideal. When $W = (T, T)$ for a Jordan triple system T, there is no need to distinguish between the two notions so we simply speak of semi-ideals. If (I^+, I^-) is an ideal of W then clearly $I^+ \cap I^-$ and $I^+ + I^-$ are ideals of T.

2.4. Lemma. *Let T be a nondegenerate Jordan triple system and $W = (T, T)$.*

(a) *W is prime if and only if T has no disjoint semi-ideals.*

(b) *If T is prime but W is not then there exist nonzero ideals I of W with $I \cap I^* = 0$. Hence, there exist polarized ideals $L = I^+ \oplus I^-$ of T such that I^+ and I^- are semi-ideals of T.*

Proof. (a) Let S_i $(i = 1, 2)$ be nonzero semi-ideals of T. Then $I_i = (S_i, P_T S_i)$ are nonzero ideals of W. Primeness of W implies $I_3 = I_1 \cap I_2 \neq 0$, and by nondegeneracy of W and 1.1 we obtain $I_3^+ = S_1 \cap S_2 \neq 0$. The converse is clear since the $+$-parts of ideals in W are semi-ideals.

(b) Since W is not prime then there exist nonzero ideals I_i $(i = 1, 2)$ of W such that $I_1 \cap I_2 = 0$. Then $I_1 \cap I_1^*$ and $I_2 \cap I_2^*$ are disjoint $*$-ideals so one must be zero since W is $*$-prime. $\square$

2.5. Definition. Let T be a Jordan triple system and $a, b \in T$. We say that a is *quadratically orthogonal in the triple sense* (q^2-orthogonal for short) to b and write $a]|[b$ or $a]|[_T b$ if $P_a P_T P_b = P_a P_T P_T P_b = 0$. A Jordan triple system T will be called *elementally prime* if it has no nonzero elements which are q^2-orthogonal. Just as for Jordan pairs (cf. Definition 1.2), it is easily seen that two orthogonal ideals of T are elementwise q^2-orthogonal, and the converse holds if T is nondegenerate. In particular, an elementally prime Jordan triple system is strongly prime.

Clearly, $a]|[b$ implies $a][b$ in (T, T) and, hence, conversely, the condition that T be elementally prime is weaker than requiring T to have no nonzero q-orthogonal elements which, by Theorem 1.10, is equivalent to (T, T) strongly prime. We will prove below (2.9) that strong primeness and elemental primeness are equivalent for Jordan triple systems. The following lemma, which is the elemental counterpart of Lemma 2.2, together with Theorem 1.10 shows that this is true at least when T is polarized.

2.6. Lemma. *A Jordan pair V is elementally prime if and only if the polarized Jordan triple system $T = V^+ \oplus V^-$ is elementally prime.*

Proof. Let V be elementally prime and let $a = a^+ \oplus a^-$, $b = b^+ \oplus b^- \in T$ be q^2-orthogonal. Since T is polarized, $P_aP_TP_b = 0$ implies $Q(a^\varepsilon)Q(V^{-\varepsilon})Q(b^\varepsilon) = 0$. By elemental primeness of V, either a^+ or b^+ is zero, and either a^- or b^- is zero. Thus if both $a^+ \neq 0 \neq a^-$ then $b = 0$. Now assume only one of a^+ and a^- is nonzero, say, $a^- = 0 \neq a^+$. Then $b^+ = 0$. Again since T is polarized, the condition $P_aP_TP_TP_b = 0$ implies $Q(a^+)Q(V^-)Q(V^+)Q(b^-)V^+ = 0$. Thus for any $c \in V^+$,

$$Q(a^+)Q(V^-)Q(Q_cb^-)V^- \subset Q(a^+)Q(V^-)Q_cQ(b^-)Q_cV^-$$
$$\subset Q(a^+)Q(V^-)Q(V^+)Q(b^-)V^+ = 0.$$

By elemental primeness of V and $a^+ \neq 0$ we obtain $Q(V^+)b^- = 0$, hence $b^- = 0$ by [9, 1.4] since V is nondegenerate.

Conversely, let T be elementally prime, and let $a^\varepsilon, b^\varepsilon \in V^\varepsilon$ be q-orthogonal. Then $P(a^\varepsilon)P_TP(b^\varepsilon) = Q(a^\varepsilon)Q(V^{-\varepsilon})Q(b^\varepsilon)V^{-\varepsilon} = 0$, and $P(a^\varepsilon)P_TP_TP(b^\varepsilon) = 0$ is automatic since T is polarized. Therefore, a^ε and b^ε are q^2-orthogonal, so one of them must be zero. This completes the proof. □

We next prove the triple version of Lemma 1.5.

2.7. Lemma. *In a nondegenerate Jordan triple system T, the relation $a]|[b$ is symmetric, and implies*

$$P_aP_TP_Tb = P_aP_TL_{T,b} = P_aP_b = P_ab = P_a\{TTb\} = L_{a,b} = 0.$$

Proof. We will make use of Lemma 1.4 for the pair (T, T). Symmetry follows from Lemmas 1.5 and 1.4(a) for $(f, f^*) = (P_aP_xP_yP_b, P_bP_yP_xP_a)$, $x, y \in T$. If we set $g = P_aP_xP_y$ we have $0 = f = gP_b$ and hence $g(b) = P_aP_xP_yb = 0$ by Lemma 1.4(b). Linearizing this with respect to y yields $P_aP_TL_{T,b} = 0$. We also have $P_aP_b = 0$ since, for any $x \in T$, P_aP_bx is trivial: $P(P_aP_bx)T = P_aP_bP_xP_bP_aT \subset P_aP_TP_TP_bT = 0$. Now $P_ab = 0$ follows from Lemma 1.4(b). For $P_a\{TTb\} = 0$ we show that $P_a\{byx\}$ is trivial for any $x, y \in T$. Indeed, by JP21,

$$P(P_a\{byx\}) = P_a P_{\{byx\}} P_a$$
$$= P_a(P_b P_y P_x + P_x P_y P_b + L_{b,y} P_x L_{y,b} - P(P_b P_y x, x)) P_a.$$

The first term vanishes by $P_a P_b = 0$, the second by $P_a P_T P_T P_b = 0$, and the fourth is, by the linearization of JP3, $P_a P(P_b P_y x, x) P_a = P(P_a P_b P_y x, P_a x) = 0$. This leaves the third term, on which we use JP12:

$$P_a(L_{b,y} P_x) L_{y,b} P_a = P_a(-P_x L_{y,b} + P(\{byx\}, x)) L_{y,b} P_a = 0$$

because $P_a P_T L_{T,b} = 0$. We finally have $L_{a,b} = 0$ since $L_{a,b} x = \{abx\}$ is trivial for any $x \in T$:

$$P(\{abx\}) = P_x P_b P_a + P_a P_b P_x + L_{x,b} P_a L_{b,x} - P(P_x P_b a, a) = 0,$$

by what we already proved and the symmetry of the relation $a]|[b$. □

2.8. Proposition. *Let T be a Jordan triple system and K a nonzero ideal of T.*

(a) *If $a, b \in K$ and T is nondegenerate then $a]|[_K b$ implies $a]|[_T b$. Hence, if T is elementally prime so is K.*

(b) *Conversely, if K is elementally prime and T is a tight extension of K then T is elementally prime. Hence, if T is prime and K is elementally prime so is T.*

Proof. If $a]|[_K b$ then in particular a and b are q-orthogonal in the ideal (K, K) of (T, T), so they are q-orthogonal in (T, T) by Proposition 1.6(a). Hence $P_a P_T P_b = 0$ and it remains to show $P_a P_T P_T P_b = 0$. This will follow from Lemma 1.4(c) applied to $f = P_a P_x P_y P_b$, $f^* = P_b P_y P_x P_a$ as soon as we show $f(K) = 0$, for any $x, y \in T$. Now use nondegeneracy: For all $c \in K$,

$$P(f(c))T = P_a P_x P_y P_b P_c P_b P_y P_x P_a T \subset P_a P_x P_y P_b P_c P_b K$$
$$\subset P_a(-P_b P_y P_x + P_{\{xyb\}} + P(P_x P_y b, b) - L_{x,y} P_b L_{y,x}) P_c P_b K$$
$$\subseteq P_a P_b K + P_a P_K P_K P_b K + P_a L_{x,y} P_b L_{y,x} P_c P_b K,$$

where we used the identity JP21. Here the first two terms vanish by Lemma 2.7 since a and b are q^2-orthogonal. On the last term we use JP12 and obtain

$$P_a(L_{x,y} P_b) L_{y,x} P_c P_b K \subset P_a(-P_b L_{y,x} + P(\{xyb\}, b)) L_{y,x} P_c P_b K$$
$$\subset P_a P_b K + P_a\{KKb\} = 0,$$

again by Lemma 2.7. This proves (a). The proof of (b) follows the same lines as that of the corresponding statements in Proposition 1.6. The details are left to the reader. □

2.9. Theorem. *A Jordan triple system is strongly prime if and only if it is elementally prime.*

Proof. If a Jordan triple system T is elementally prime then it is strongly prime, as noted in Definition 2.5. Conversely, assume that T is strongly prime. There are two cases.

Case 1: (T, T) is prime. Then it is elementally prime by Theorem 1.10, so a fortiori T is elementally prime.

Case 2. (T, T) is not prime. Then by Lemma 2.4(b), T contains a nonzero polarized ideal $L = I^+ \oplus I^-$. By [11] L inherits strong primeness, so by Lemma 2.2 the Jordan pair (I^+, I^-) is strongly prime as well. By Theorem 1.10 it is elementally prime, which implies L elementally prime by Lemma 2.6. Now Proposition 2.8(b) shows that T is elementally prime, as required. □

Notice how we reduce the problem in the triple case to the pair case by either going up to (T, T) or down to a polarized ideal, which again corresponds to a pair.

3. Local algebras and strong primeness of Jordan systems

3.1. We now study the relationship of strong primeness of a Jordan system and strong primeness of its local algebras and subquotients. Let V be a Jordan pair and let $b \in V^-$. Then V^+ becomes a Jordan algebra $V^+_{(b)}$, the homotope at b, with quadratic operators $U_x y = Q_x Q_b y$ and squaring $x^2 = Q_x b$ [7, 1.9], and $\operatorname{Ker} b = \{x \in V^+ \colon Q_b x = Q_b Q_x b = 0\}$ is an ideal of $V^+_{(b)}$ [7, 4.19]. If V is nondegenerate then $\operatorname{Ker} b = \operatorname{Ker} Q_b$ [7, 4.20]. The *local algebra* of V at b is $V_b^+ = V^+_{(b)}/\operatorname{Ker} b$. Local algebras at elements of V^+ are defined by switching the roles of $+$ and $-$. See [5] for an exhaustive study of these algebras.

Given an inner ideal $N \subset V^-$, the *subquotient* of V with respect to N [9] is the Jordan pair $S = (V^+/\operatorname{Ker} N, N)$ where $\operatorname{Ker} N = \{x \in V^+ \colon Q_N x = Q_N Q_x N = 0\}$. Subquotients with respect to inner ideals contained in V^+ are defined analogously. If $b \in N \subset V^-$, then it is easy to see that the local algebras V_b^+ and S_b^+ are canonically isomorphic.

The following theorem gives a positive answer to [9, 2.8].

3.2. Theorem. *The following conditions on a Jordan pair V are equivalent.*

(i) *V is strongly prime;*

(ii) *all local algebras of V at nonzero elements are strongly prime;*

(iii) *V is nondegenerate and V_b^+ is strongly prime for all $0 \neq b \in V^-$;*

(iv) *V is nondegenerate and V_a^- is strongly prime for all $0 \neq a \in V^+$;*

(v) *all nonzero subquotients of V are strongly prime;*

(vi) *V is nondegenerate and all subquotients determined by nonzero inner ideals in V^+ are strongly prime;*

(vii) *V is nondegenerate and all subquotients determined by nonzero inner ideals in V^- are strongly prime.*

Proof. (i) $\Rightarrow$ (ii): Since V is strongly prime if and only if V^{op} is, it suffices to consider a local algebra $J = V_b^+$. By Theorem 1.13 we are reduced to showing that J is elementally prime. Let $x \mapsto \bar{x} = x + \operatorname{Ker} b$ denote the canonical homomorphism $V^+ \to J$ and assume $\bar{x}$ is q-orthogonal to $\bar{y}$ in J, i.e., $U_{\bar{x}} U_J U_{\bar{y}} = 0$. This implies $Z = Q_x Q_b Q(V^+) Q_b Q_y Q_b V^+ \subset \operatorname{Ker} b$ whence $0 = Q_b Z = Q(Q_b x) Q(V^+) Q(Q_b y) V^+$. By elemental primeness of V (Theorem 1.10) we have either $Q_b x = 0$ or $Q_b y = 0$, hence either $x \in \operatorname{Ker} b$ or $y \in \operatorname{Ker} b$ by nondegeneracy of V. This means either $\bar{x} = 0$ or $\bar{y} = 0$, as desired.

The implications (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are evident. Let us prove (iii) $\Rightarrow$ (i). We show that V has no nonzero orthogonal ideals. If I and L are orthogonal ideals of V then $(I^+ + \operatorname{Ker} b)/\operatorname{Ker} b$ and $(L^+ + \operatorname{Ker} b)/\operatorname{Ker} b$ are orthogonal ideals of V_b^+ for any $b \in V^-$. Since all V_b^+ are prime we have either $I^+ \subset \operatorname{Ker} b$ or $L^+ \subset \operatorname{Ker} b$ for any $b \in V^-$, hence either $Q_b I^+ = 0$ or $Q_b L^+ = 0$. Since V is nondegenerate, it follows from [11, 1.7] that V^- is contained in the union of the annihilators of I and L and therefore in one of them. If, for example, V^- is contained in the annihilator of I then $Q(V^-) I^+ = 0$, hence $I^+ = 0$ by [9, 1.4] since V is nondegenerate, and $I = 0$ by 1.1. This proves (iii) $\Rightarrow$ (i), and (iv) $\Rightarrow$ (i) follows by passing to V^{op}.

For the remaining implications, consider a subquotient $S = (V^+/\operatorname{Ker} N, N)$ which inherits nondegeneracy from V by [9, 2.2]. By applying the equivalence of (i) and (iii) to S in place of V and using $V_b^+ = S_b^+$ for all $b \in N$, as noted in 3.1, we see (i) $\Rightarrow$ (vii). Now (i) $\Rightarrow$ (vi) follows by passing to V^{op}, and therefore (i) $\Rightarrow$ (v) holds as well. The implications (vi) $\Rightarrow$ (i) and (vii) $\Rightarrow$ (i) are clear since $\operatorname{Ker} V^{\varepsilon} = 0$ by nondegeneracy [9, 1.4], so that V equals the subquotient determined by the inner ideal V^{ε}. Finally, if (v) holds then $(V^+, V^-/\operatorname{Ker} V^+)$ is in particular nondegenerate so that V^+ has no trivial elements. Similarly, neither does V^- so V is nondegenerate, and we have the missing implications (v) $\Rightarrow$ (vi) and (v) $\Rightarrow$ (vii). $\square$

Applying this to the Jordan pair of a Jordan algebra, together with Lemma 1.12 and the fact that the local algebras of a Jordan algebra and its associated Jordan pair are obviously the same, we obtain the following result which contains [1, (4.1) (iii)].

3.3. Corollary. *A Jordan algebra is strongly prime if and only if all its local algebras at nonzero elements are strongly prime.*

We consider next the local characterization of primeness for a Jordan triple system. Note that the case where the associated Jordan pair is prime is covered by Theorem 3.2. To deal with the remaining case we need a lemma.

3.4. Lemma. *Let T be a strongly prime Jordan triple system with $W = (T, T)$ not prime. Then W is a subdirect product of a strongly prime Jordan pair and its opposite.*

Proof. By Lemma 2.4(b) W contains nonzero ideals $I = (I^+, I^-)$ with $I \cap I^* = 0$, equivalently, $I^+ \cap I^- = 0$. We use Zorn's Lemma to find a maximal such ideal M.

Since $M \cap M^* = 0$ the canonical map $W \to (W/M) \oplus (W/M^*)$ is injective, and clearly $W/M^* \cong (W/M)^{\mathrm{op}}$. Thus it remains to show that $V = W/M$ is strongly prime. To see that V is nondegenerate, let rad $V = L/M$ be the lower radical of V. We claim that $L^+ \cap L^- \subset \operatorname{rad} T = 0$. Indeed, let $x \in L^+ \cap L^-$. For any m-sequence (x_i) in T beginning with x, we have two m-sequences $x_i + M^\varepsilon \in V^\varepsilon$, $\varepsilon = \pm$, in V, beginning with $x + M^\varepsilon$. By the characterization of the lower radical in terms of m-sequences [12], both of these must terminate. This allows us to find an n_0 such that $x_i \in M^+ \cap M^- = 0$ for all $i > n_0$. Therefore, all m-sequences of T beginning with x terminate, which means $x \in \operatorname{rad} T$ by the characterization of the lower radical mentioned above, and $\operatorname{rad} T = 0$ by strong primeness. Since M is maximal with the property that $M^+ \cap M = 0$ we have $L = M$ and hence rad $V = 0$.

Next, we show any two nonzero ideals I_1/M and I_2/M of V have nonzero intersection. Suppose to the contrary that $I_1 \cap I_2 \subset M$. Then $J_1 = I_1 \cap I_1^* \neq 0 \neq I_2 \cap I_2^* = J_2$ (by maximality of M) are nonzero $*$-ideals of W and hence $J_1 \cap J_2 \neq 0$ by $*$-primeness of W. But $J_1 \cap J_2 = (I_1 \cap I_2) \cap (I_1^* \cap I_2^*) \subset M \cap M^* = 0$, a contradiction. Now it follows that W/M is strongly prime and the proof is complete. □

3.5. Theorem *Let T be a strongly prime Jordan triple system and $W = (T, T)$.*

(a) *If W is prime then all local algebras of T at nonzero elements are strongly prime.*

(b) *If W is not prime then for any $0 \neq b \in T$, the local algebra T_b is either strongly prime or a subdirect product of two strongly prime Jordan algebras. Moreover, for all ideals $I = (I^+, I^-)$ of W with $I \cap I^* = 0$ (which always exist by Lemma 2.4(b)) and all $0 \neq b \in I^\varepsilon$, the local algebra T_b is strongly prime.*

Proof. (a) is immediate from Theorem 3.2.

(b) By Lemma 3.4 W is a subdirect product of two strongly prime Jordan pairs W_1 and W_2. Hence, the local algebras $T_b = W_b^+$ are isomorphic to the subdirect product of local algebras of W_1 and W_2 [5, 1.3.6], and these are strongly prime by Theorem 3.2 or zero.

Now let I be an ideal with the properties stated above, let $b \in I^-$ and denote by $\bar{x} = x + \operatorname{Ker} b \in J = T_b$ the canonical image of $x \in T$. We show that J is elementally prime. If $\bar{x}$ and $\bar{y}$ are q-orthogonal in J then $Z = P_x P_b P_T P_b P_y P_b T \subset \operatorname{Ker} b$, hence $0 = P_b Z = P(P_b x) P_T P(P_b y) T$. Now $u = P_b x$ and $v = P_b y$ belong to I^- since I^- is in particular an inner ideal of T, and we have $P_u P(I^+) P_v I^+ = 0$. Therefore, u is q-orthogonal to v in the Jordan pair I. Since $I^+ \oplus I^-$ inherits strong primeness from T [11] and hence I is elementally prime by Lemma 2.2 and Theorem 1.10, we conclude $u = 0$ or $v = 0$. By nondegeneracy, either $\bar{x} = 0$ or $\bar{y} = 0$ so J is elementally prime.

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